Pioneering Applications of Fractional Differential Equations in Complex Systems

SYMMETRY 2024

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Pioneering Applications of Fractional Differential Equations in Complex Systems

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Can integer order differential equations be used to solve real-life phenomena?

How to find the exact solutions of inhomogeneous fractional diffusion equations?

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What are Fractional diffusion equations?

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Pioneering Applications of FDEs

- Control Theory in Engineering
- Signal Processing
- Biological Systems
- Physics
- Finance and Economics
- Geophysics and Seismology
- Fluid Mechanics

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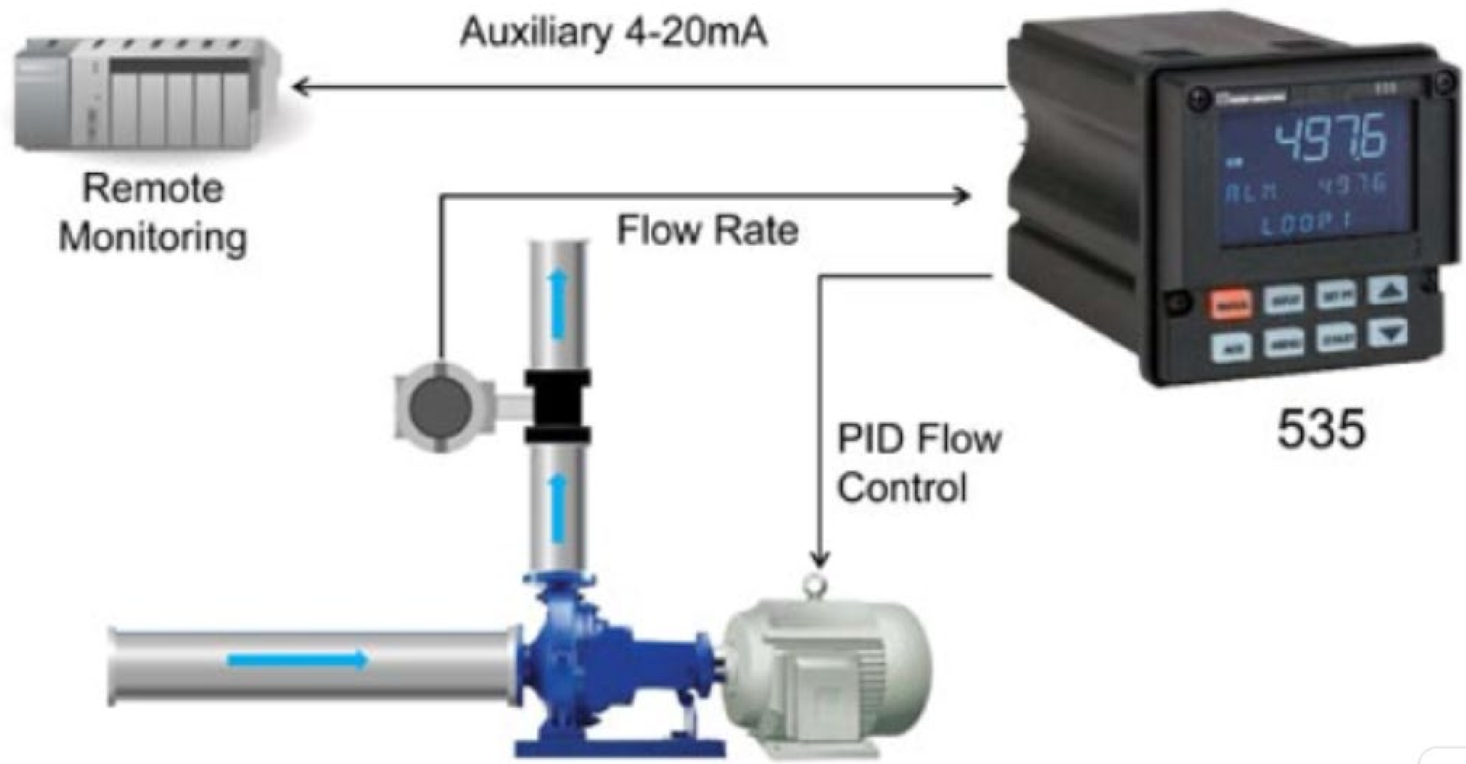
1. Control Theory and Engineering



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Example

Fractional optimal control

$$\min J(x,u) = \int_{t_0}^{T} L(t, x(t), u(t,x)dt \qquad \ldots(1)$$

s.t.

$$k_1 \frac{\partial x}{\partial t} + k_2\, D_{t_0^+}^{\alpha} x(t) = f\big(t, x(t), u(t,x)\big), \quad t_0 < t < T \qquad \ldots(2)$$

$$x(t_0) = x_0, \qquad x(t_f) = x_f \qquad \ldots(3)$$

$$u_{min} \le u(t,x) \le u_{max}, \quad t_0 \le t \le T \qquad \ldots(4)$$

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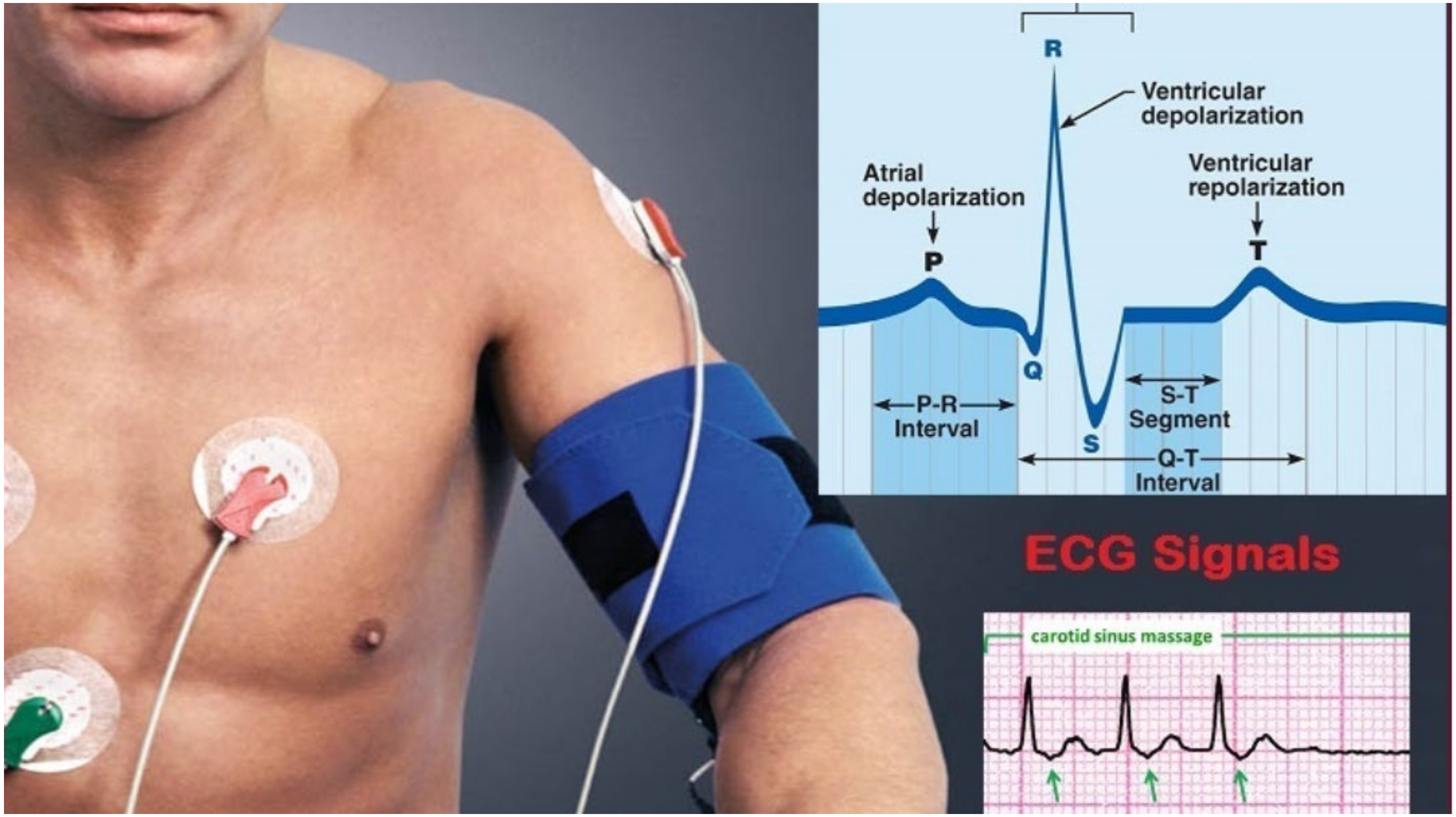
2. **Signal Processing**



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Example

Consider the standard diffusion equation for signal problem:

- Dependent on time variable $\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial t^2}$
- Dependent on space variable $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

With the initial condition $u(t,0) \ = \ s(t)$,

the solution is

$$u(t,x) = \frac{1}{\sqrt{4\pi x}}\int_{-\infty}^{\infty} e^{-\frac{(t-?)^2}{4kb}} s(\xi)d\xi$$

which is a Gaussian blurs and is used in signal processing for reducing Gaussian noises.

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Diffusion equation

$$\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial t^2}$$

⇨

Fractional signal smoothing equation

$$\frac{\partial u}{\partial x} = k_p \, D_t^{\alpha_p} \, u(t,x)$$

where the Caputo fractional derivative of order $1 < \alpha_p \leq 2$ is defined by

$$D_t^{\alpha_p} u(x,t) = \frac{1}{\Gamma(2-\alpha_p)} \int_{t_0}^{t} \frac{u_{tt}(x,s)}{(t-s)^{\alpha_p - 1}} ds$$

Diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

⇨

Fractional signal smoothing equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^\alpha u}{\partial |x|^\alpha}$$

$$\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)}\left(D_{x+}^{\alpha} u + D_{x-}^{\alpha} u\right)$$

- $D_{x+}^{\alpha} u = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_{x_0}^{x} \frac{u(s,t)}{(x-s)^{\alpha-1}} ds$
- $D_{x-}^{\alpha} u = \frac{-1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_{x}^{X_{max}} \frac{u(s.t)}{(s-x)^{\alpha-1}} ds$

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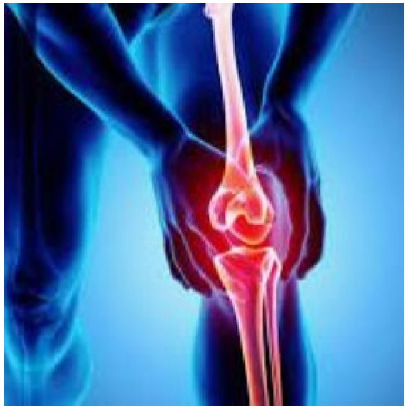
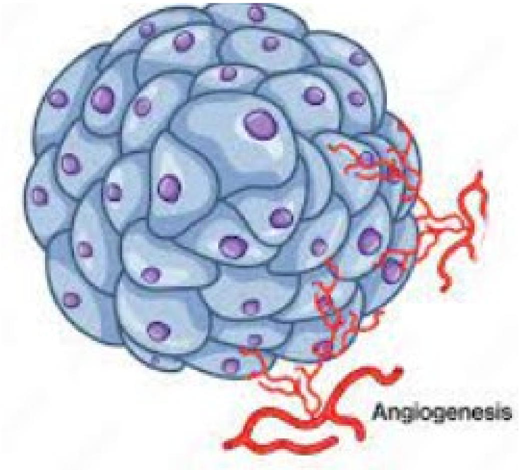
3. Biological Systems





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Let v be the tumor volume

a present the growth exponent (the kinetic parameter)

$$\frac{dv}{dt} = f \quad \Rightarrow \quad \frac{d^{\alpha}v}{dt^{\alpha}} = f$$

$$f = \begin{cases} a\,v(t) & \text{fractional order exponential model} \\ a\,v(t)\left(1 - \left(\frac{v(t)}{k}\right)^2\right) & \text{fractional order logistic model} \\ a\,v(t)\ln\left(\frac{b}{v(t)+c}\right) & \text{fractional order Gompertz model} \\ \begin{cases} p\,v^a - q\,v^b; \;\; a \neq b \\ p\,v^a - \ln(v)\,q\,v^b; \;\; a = b \end{cases} & \text{fractional order Bertalanffy-Putter model} \end{cases}$$

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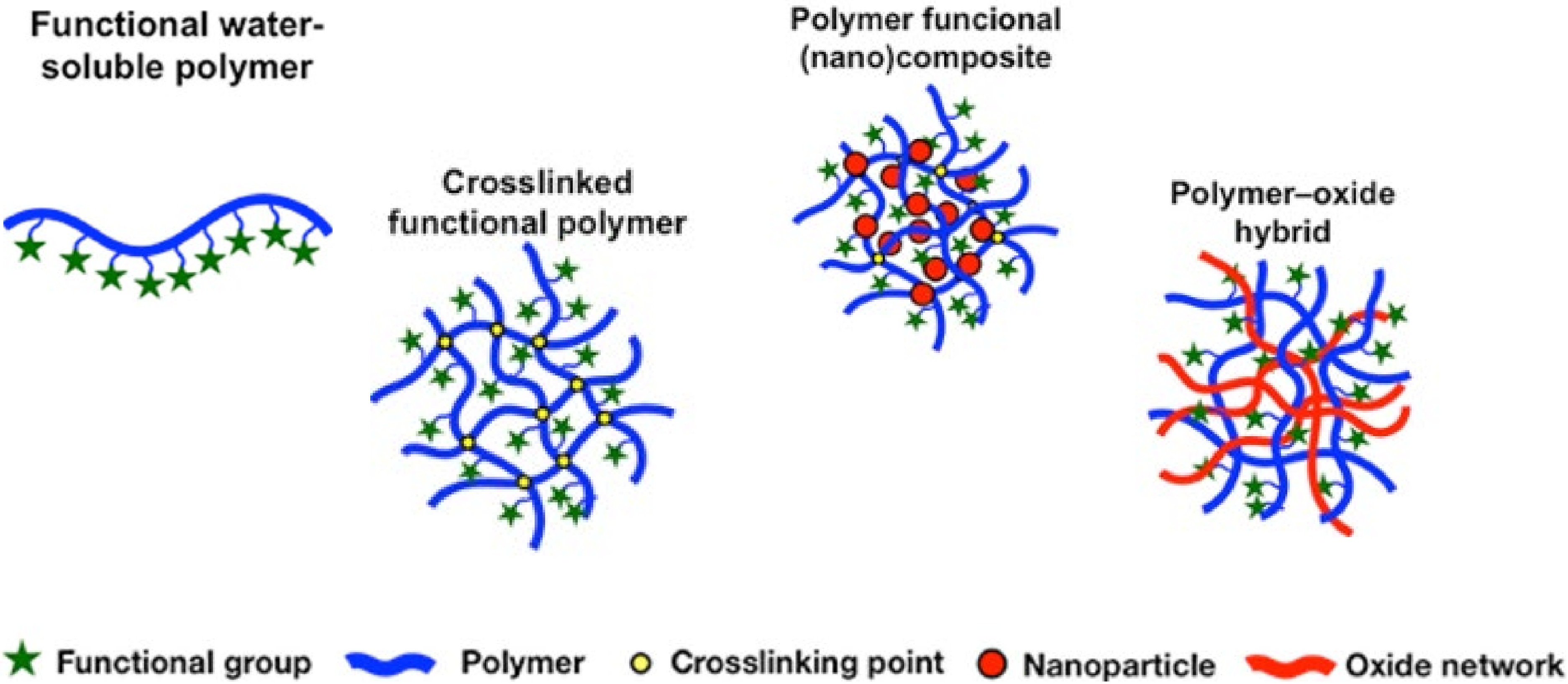
4. Physics



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According to the principle of conservation of mass, the equation of continuity form is given by

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = f(x,t)$$

$$Q(x,t) = -C(x)\frac{\partial}{\partial x}\int_a^x K_+(x,\xi)u(\xi,t)\,d\xi - D(x)\frac{\partial}{\partial x}\int_x^b K_-(x,\xi)u(\xi,t)\,d\xi$$

with
$$\begin{cases} K_+(x,\xi) = \frac{1}{\Gamma(1-\alpha)}(x-\xi)^{-\alpha}, & a \le \xi \le x; \\ K_-(x,\xi) = \frac{1}{\Gamma(1-\alpha)}(\xi-x)^{-\alpha}, & x \le \xi \le b, \end{cases}$$

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- A one-dimensional two-sided space-fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(C(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} - D(x)\frac{\partial^{\alpha} u(x,t)}{\partial (-x)^{\alpha}} \right) + f(x,t), \quad a \le x \le b, 0 < \alpha < 1, t > 0$$

- A two-dimensional two-sided space-fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left(C_x(x,y)\frac{\partial^{\alpha} u(x,y,t)}{\partial x^{\alpha}} - D_x(x,y)\frac{\partial^{\alpha} u(x,y,t)}{\partial (-x)^{\alpha}} \right) + \frac{\partial}{\partial y}\left(C_y(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial y^{\beta}} - D_y(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial (-y)^{\beta}} \right) + f(x,y,t), \quad (x,y) \in \Omega, t > 0$$

with the left and right Riemann–Liouville fractional derivatives

$$\begin{cases} \dfrac{\partial^{\alpha} u}{\partial x^{\alpha}} = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial x} \displaystyle\int_a^x \dfrac{u(s,y,t)}{(x-s)^{\alpha}} ds; \\ \dfrac{\partial^{\alpha} u}{\partial (-x)^{\alpha}} = \dfrac{-1}{\Gamma(1-\alpha)} \dfrac{\partial}{\partial x} \displaystyle\int_x^b \dfrac{u(s,y,t)}{(s-x)^{\alpha}} ds; \end{cases}$$

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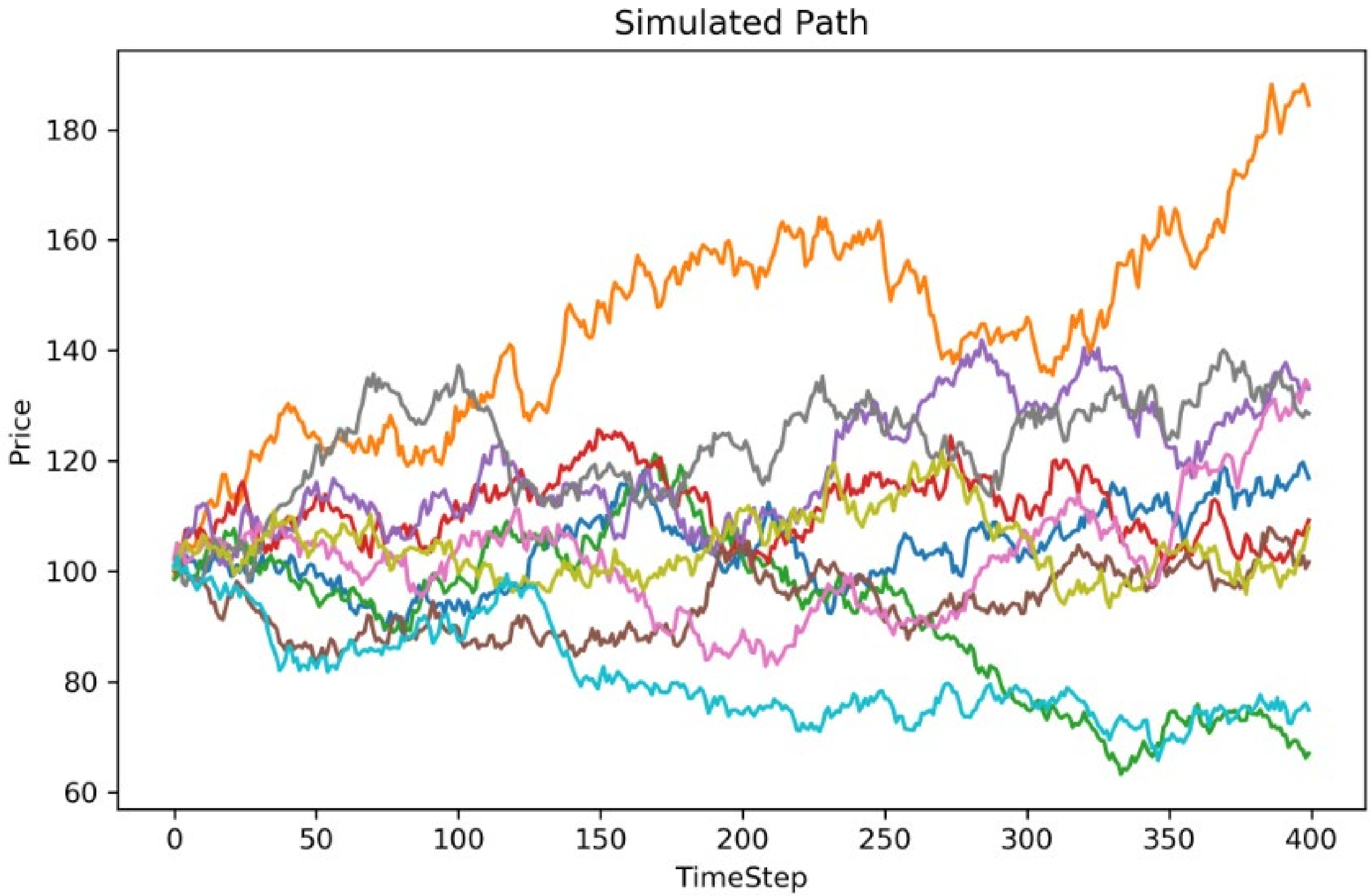
5. Finance and Economics



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- The stochastic DE of the Black-Sholes model for pricing of a European call option on a non dividend paying stock:

$$S_t = S_0 + \int_0^t r S_\tau d\tau + \int_0^t \sigma S_\tau dB_\tau$$

- The PDE of a European call option :

$$\frac{\partial V(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S_t,t)}{\partial S_t^2} + rS_t \frac{\partial V(S_t,t)}{\partial S_t} - rV(S_t,t) = 0,$$

with terminal payoff $\mathrm{V}(S_t\,,\mathrm{T}) = \psi(S_t)$

Closed-form solutions

$$C(S_t,t) = G(d_1)S - G(d_2)Ke^{-r(T-t)},$$
$$P(S,t) = Ke^{-r(T-t)} - S + C(S,t),$$
$$= G(-d_2)Ke^{-r(T-t)} - G(-d_1)S,$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right],$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

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A European option with terminal payoff $\psi(S_t)$

$$\frac{\partial P(S,t)}{\partial t} + rS\frac{\partial P(S,t)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 P(S,t)}{\partial S^2} - rP(S,t) = 0$$

$$\Downarrow$$

$$\frac{\partial P(S,t)}{\partial t} + rS\frac{\partial P(S,t)}{\partial S} + \frac{\sigma^2 S^2}{2} D_{-\infty}^{\alpha} P(S,t) - rP(S,t) = 0$$

$D_{-\infty}^{\alpha}$ is the left-side RiemannLiouville (RL) fractional derivative

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Riemann-Liouville fractional derivatives

1. **Fractional Integral:**

   The fractional integral of order $\alpha > 0$ of a function $f(t)$ is defined as:

   $$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

   where $\Gamma(\alpha)$ is the Gamma function.

2. **Fractional Derivative:**

   The Riemann-Liouville fractional derivative of order α, where $n-1 < \alpha < n$ and n is an integer, is defined using the fractional integral. It is given by:

   $$(D^\alpha f)(t) = \frac{d^n}{dt^n} (I^{n-\alpha} f)(t)$$

   Here, $\frac{d^n}{dt^n}$ represents the ordinary n-th derivative.

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6. **Geophysics and Seismology**

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The wave equation

$$\nabla\sigma(t) = \rho \frac{d^2u}{dt^2} + f \quad \Rightarrow$$

Fractional spatial derivatives

$$\nabla^2\left(1 + \alpha\tau^\alpha \frac{d^\alpha}{dt^\alpha}\right)u(t) + \frac{1}{c^2}\frac{d^2u}{dt^2}$$

$$c = \sqrt{E/\rho}$$

s.t. BCs: $u(0,t) = u(L,t) = 0, t \geq 0.$

ICs: $\frac{\partial u(x,0)}{\partial t} = \frac{\partial u(x,0)}{\partial t} = 0, \; 0 \leq x \leq L$

$$u(x,0) = \varphi(x), \qquad 0 \leq x \leq L$$

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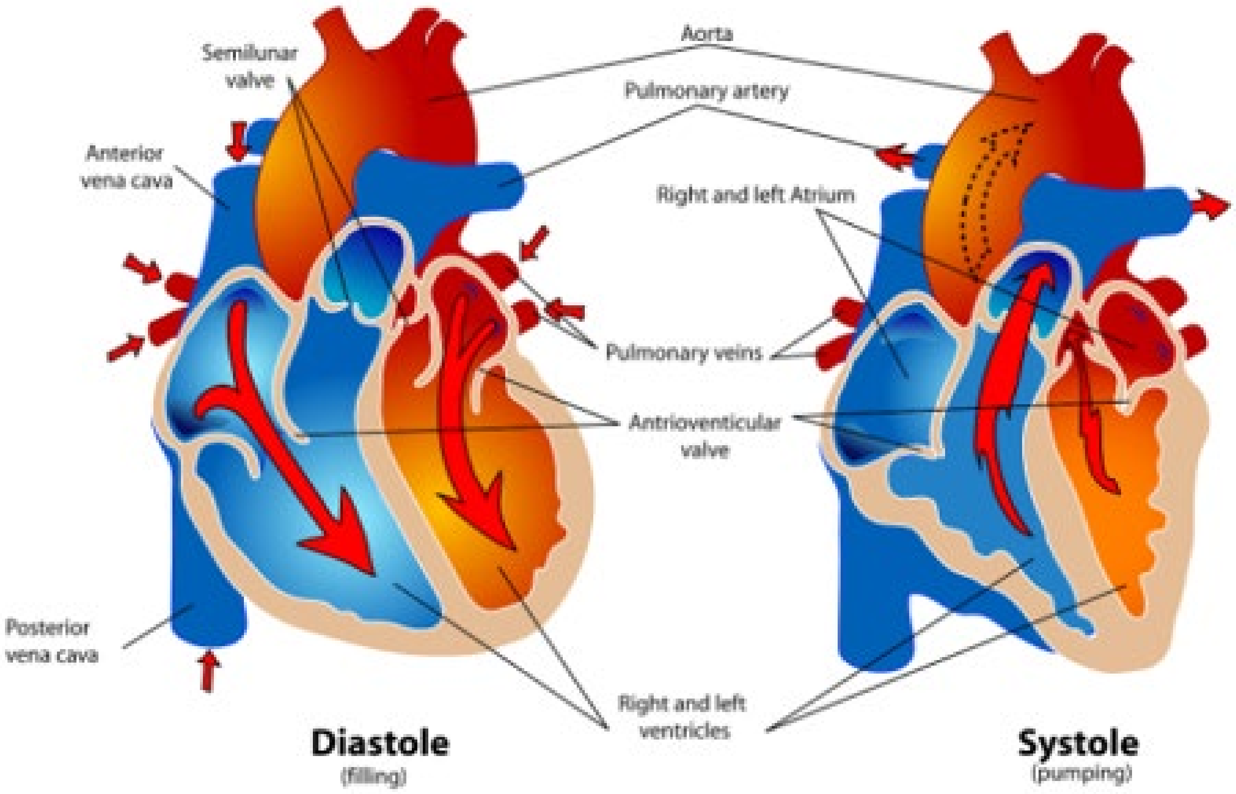
7. Fluid Mechanics



Blood flow problem

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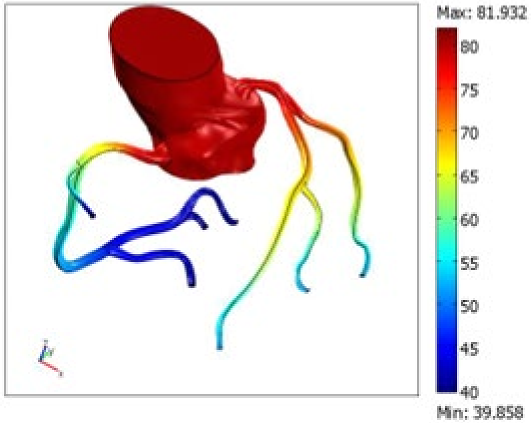

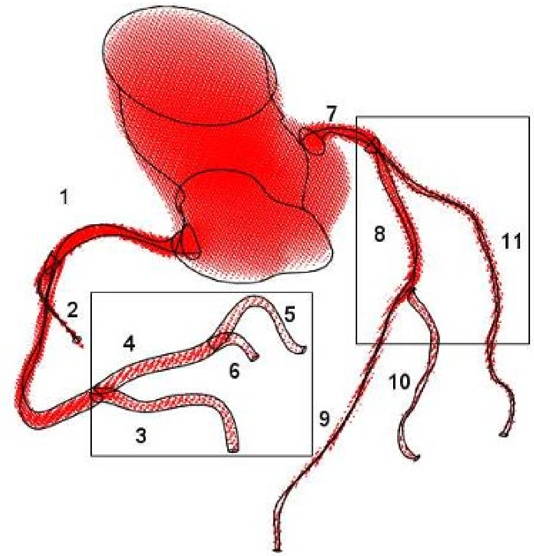

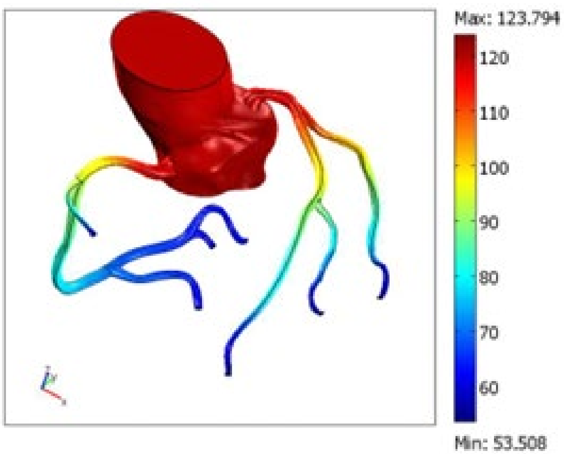

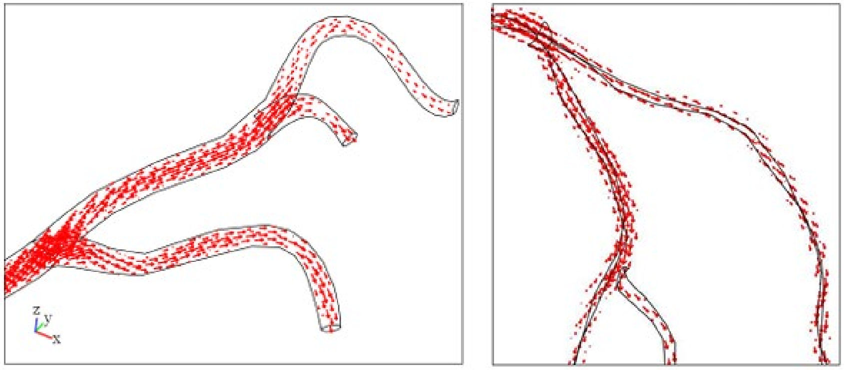

(a) (b)

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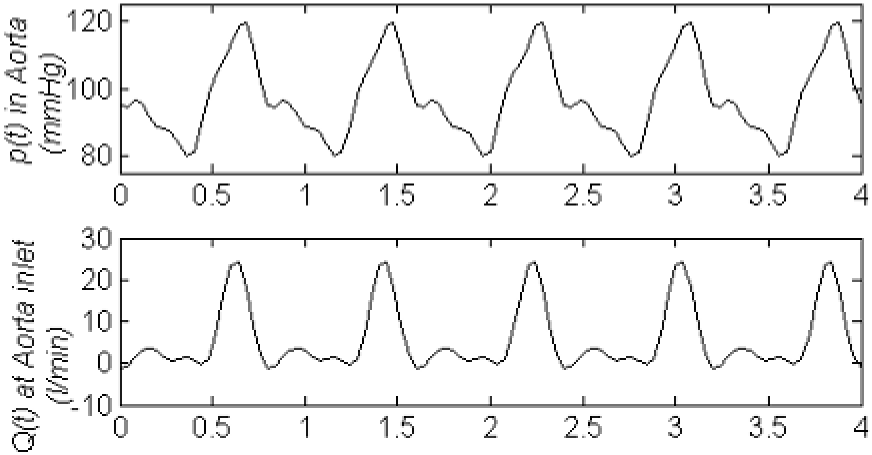
$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_1, \\ \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \dfrac{1}{\rho} \nabla \cdot \sigma \quad \text{in } \Omega_1, \end{cases}$$

$$\begin{cases} \sigma = -pI + 2\eta(\dot{\gamma})\mathbf{D}, \\ \mathbf{D} = \dfrac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \end{cases}$$

BCs:

$$\bar{u}_{in}(t) = Q(t)/A$$

$$\sigma \cdot \mathbf{n} = -p(t)\mathbf{n}$$



Fractional Navier-Stokes equations:

$$D_t^{\alpha}\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{1}{\rho}\nabla \cdot \sigma$$

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Freeway Traffic Flow Prblem

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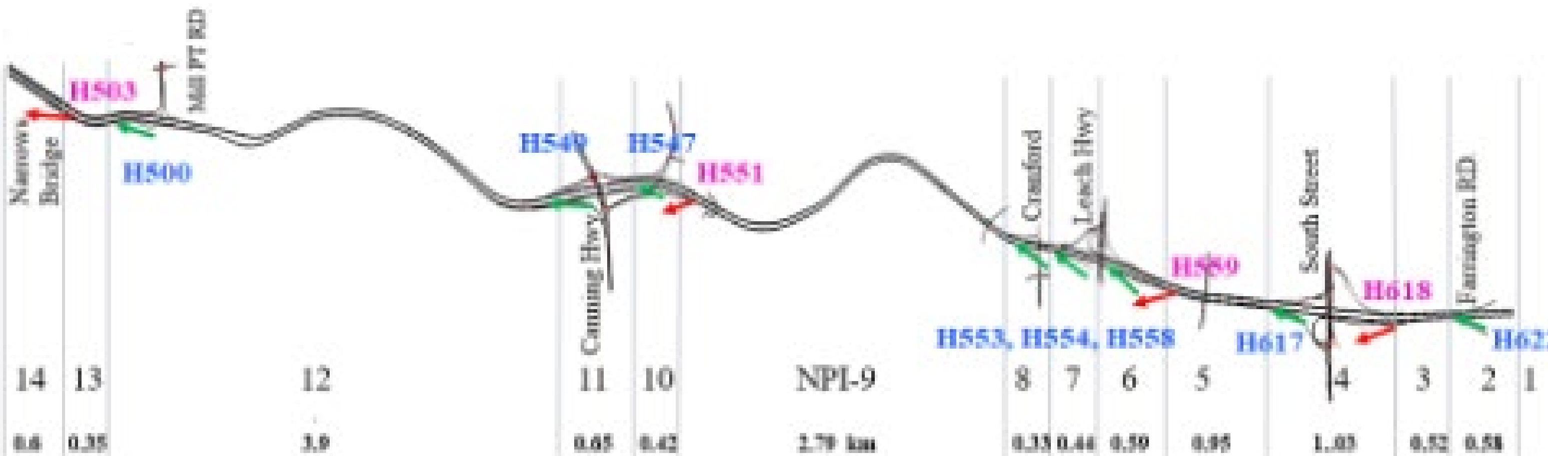

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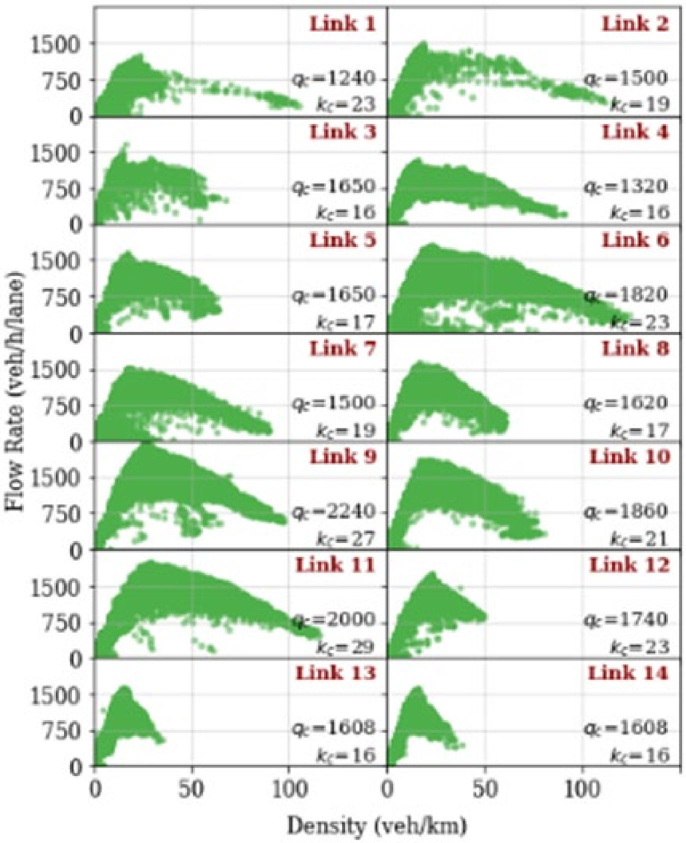
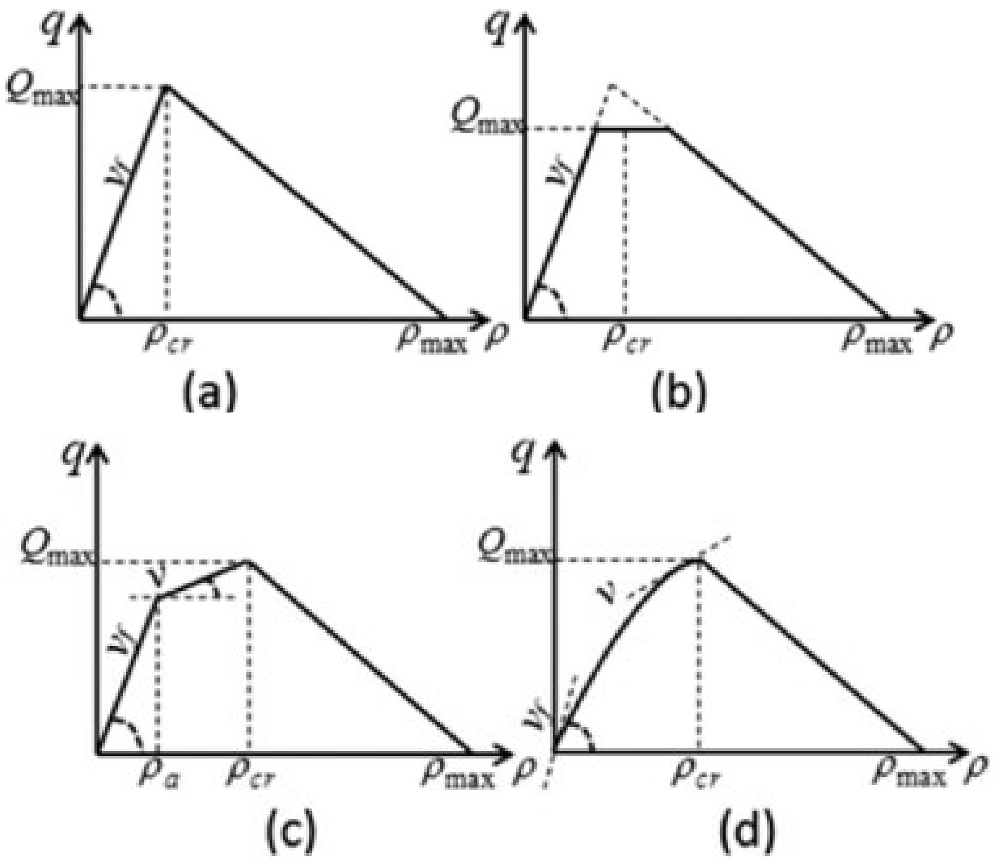
Fundamental Diagram of Traffic Q - ρ Relations





(a) a triangular FD (CTM), (b) a trapezoidal FD, (c) a piecewise linear FD and (d) a nonlinear FD.

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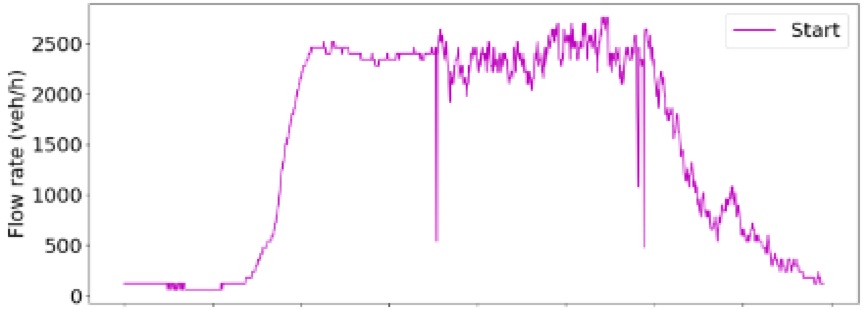
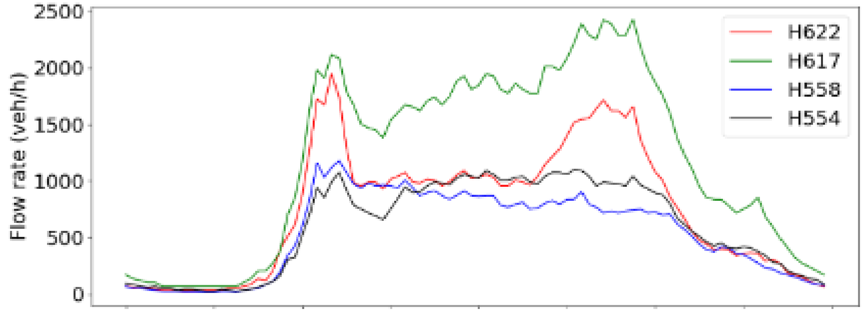
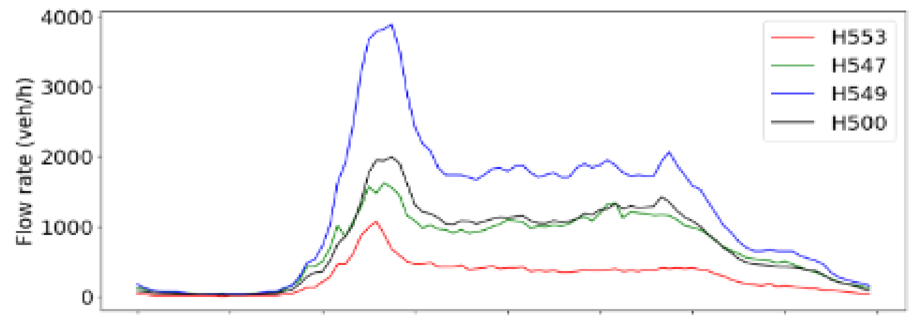
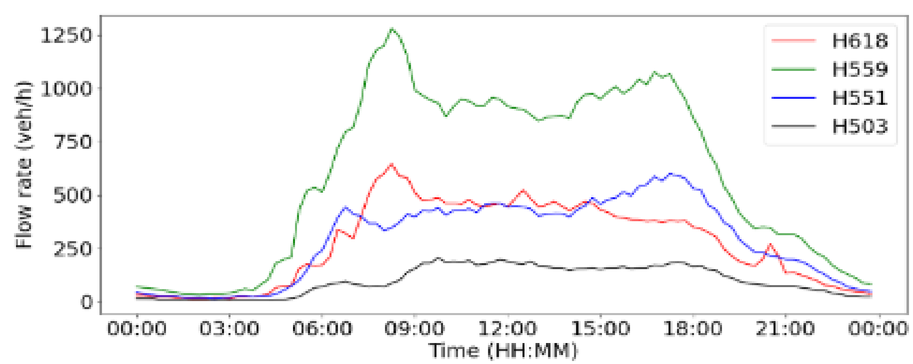


The arrival rates at the beginning.



The arrival rates at 8 on-ramps





The discharge rates at 4 off-ramps

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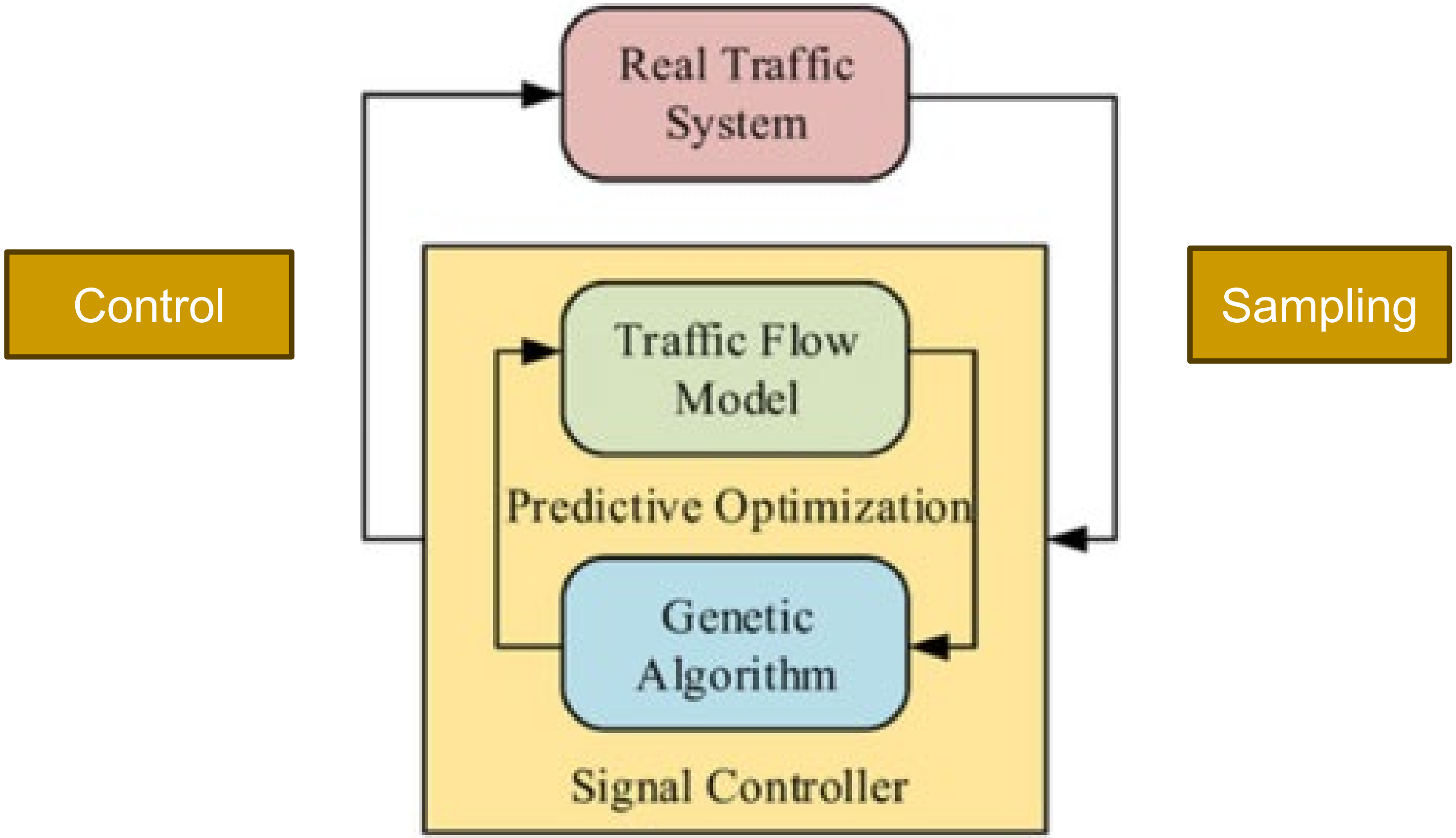

Traffic flow management

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$$\min \sqrt{\frac{1}{m}\sum_{t=0}^{m\Delta t}\left(\sum_{i=1}^{n}(\rho_i(t) - \hat{\rho}_i(t))^2\right)}$$

s.t.

$$\frac{\partial^\alpha \boldsymbol{\rho}}{\partial t^\alpha} + \frac{\partial \boldsymbol{q}}{\partial x} = \frac{\partial^\beta}{\partial x^\beta}\left(D(x)\frac{\partial \boldsymbol{\rho}}{\partial x}\right) - \frac{1}{\tau}\boldsymbol{\rho} + S(x,t)$$

$$\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_n)^T; \ \boldsymbol{q} = (q_1, q_2, \dots, q_n)^T$$

$$0 \le \rho_i \le \rho_{jam}; 0 \le q_i \le q_{max}$$

and

$$q_i(t) = \min\{q_{i-1}(t), q_{max}(t), w_i(t)(\rho_{jam} - \rho_{i+1}(t))\}$$

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Numerical approximations

For the time derivative $0 < \alpha < 1$:

$$\frac{\partial^\alpha \boldsymbol{\rho}}{\partial t^\alpha} \approx \frac{1}{\Delta t^\alpha} \sum_{k=0}^{m} \binom{\alpha}{k} \boldsymbol{\rho}(x, t - (m-k)\Delta t)$$

For second-order space derivative $0 < \beta < 1$:

$$\frac{\partial^\beta}{\partial x^\beta}\left(D(x) \frac{\partial \boldsymbol{\rho}}{\partial x} \right) \approx \frac{1}{\Delta x^\beta} \sum_{k=0}^{n} \binom{\beta}{k} D(x + (n-k)\Delta x) \frac{\partial}{\partial x} \boldsymbol{\rho}(x + (n-k)\Delta x, t)$$

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$$\alpha_i(t) = \begin{cases} \alpha_{i,1} & \text{if} \quad t \in [t_0,\ t_0 + \Delta t] \\ \alpha_{i,2} & \text{if} \quad t \in [t_0 + \Delta t,\ t_0 + 2\Delta t] \\ & \vdots \\ \alpha_{i,m} & \text{if} \quad t \in [t_0 + (m-1)\Delta t,\ t_0 + m\Delta t], \end{cases}$$

$$\beta_i(t) = \begin{cases} \beta_{i,1} & \text{if} \quad t \in [t_0,\ t_0 + \Delta t] \\ \beta_{i,2} & \text{if} \quad t \in [t_0 + \Delta t,\ t_0 + 2\Delta t] \\ & \vdots \\ \beta_{i,m} & \text{if} \quad t \in [t_0 + (m-1)\Delta t,\ t_0 + m\Delta t], \end{cases}$$

$$0 < \alpha_{i,t} < 1$$

$$0 < \beta_{i,t} < 1$$

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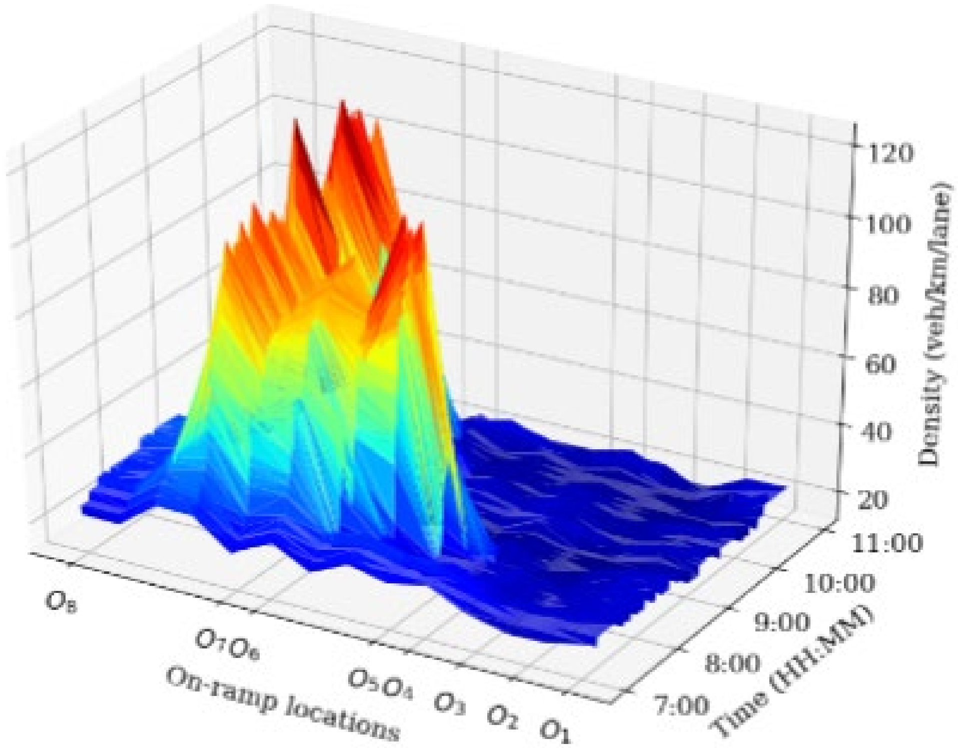

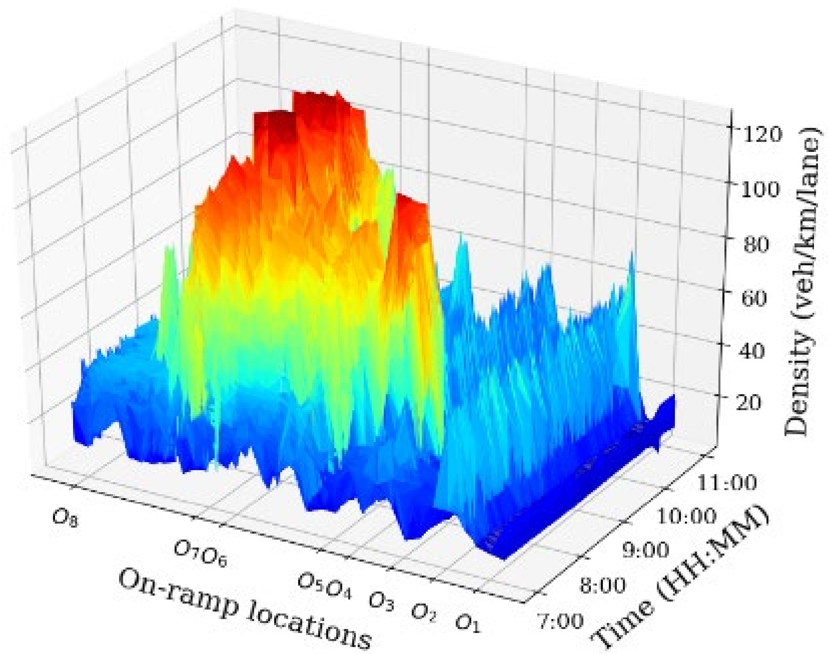

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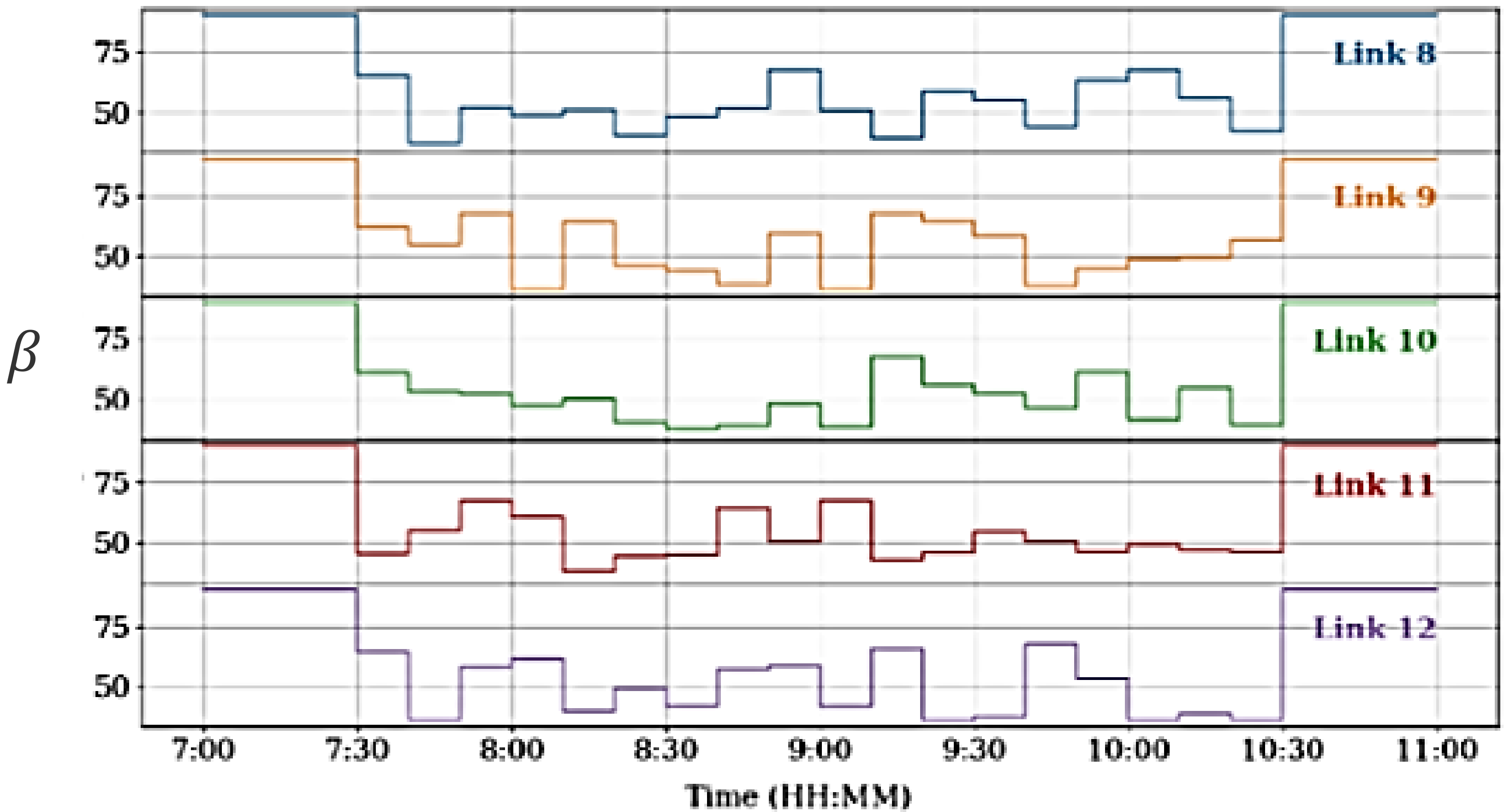

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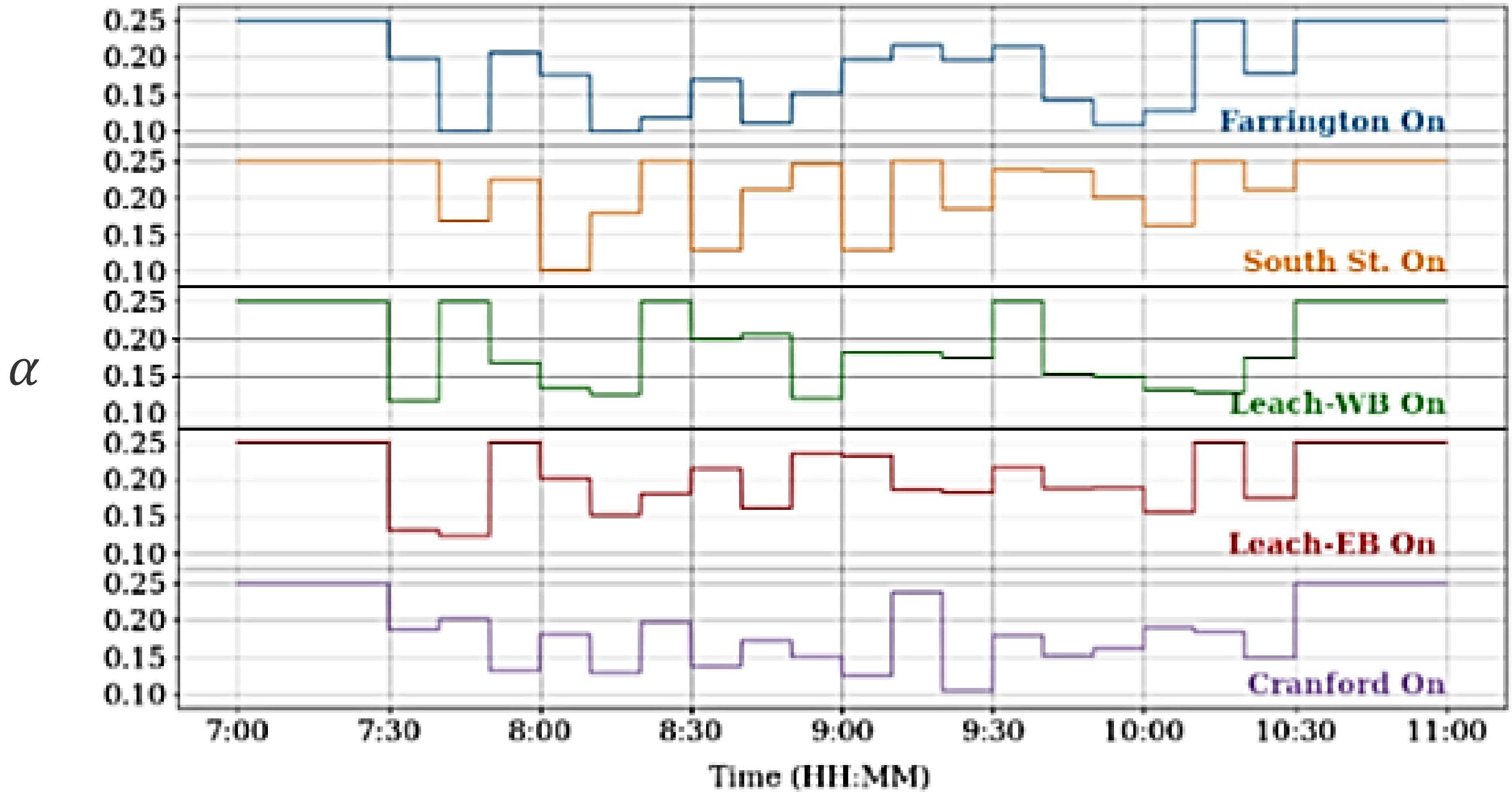

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We have derived a closed-form solution for a scenario where there is no incoming flow from the onramp and offramp, ensuring that the flow entering and exiting the link is balanced., $\alpha = 1/2, \beta = 1$ and $\tau = 1\ second$.

$$\frac{\partial^{1/2}\boldsymbol{\rho}}{\partial t^{1/2}} = \frac{\partial}{\partial x}\left(D(x)\frac{\partial\boldsymbol{\rho}}{\partial x}\right) - \frac{1}{\tau}\boldsymbol{\rho}$$

- Assume a separation solution of the form $\rho(x,t) = X(x)T(t)$.
- Substituting this into the equation gives:

$$\frac{T''(t)}{T(t)} = \frac{D(x)X''(x)}{X(x)} - 1$$

Divide both sides by -1 to make the equation more standard:

$$-\frac{T''(t)}{T(t)} = 1 - \frac{D(x)X''(x)}{X(x)}$$

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Step 2: Solve the Time Component

The left-hand side involves a fractional derivative in time ($\frac{T''(t)}{T(t)}$), which corresponds to a fractional ordinary differential equation. We can solve this equation with a fractional order derivative.

Assume $\frac{d^\alpha T(t)}{dt^\alpha} = \lambda^2 T(t)$, where λ^2 is a constant to be determined. The solution to this fractional differential equation can be expressed as:

$$T(t) = C_1 E_\alpha(\lambda t^\alpha) + C_2 E_\alpha(-\lambda t^\alpha)$$

Where C_1 and C_2 are constants, and $E_\alpha(z)$ is the Mittag-Leffler function of order α.

Step 3: Solve the Space Component

The right-hand side involves a spatial diffusion term with $D(x)X''(x)/X(x)$, which corresponds to a second-order ordinary differential equation in space. We can solve this equation separately.

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Assume $\frac{d^2X(x)}{dx^2} + k^2X(x) = 0$, where k^2 is a constant. The solution to this ordinary differential equation is:

$$X(x) = C_3\cos(kx) + C_4\sin(kx)$$

Step 4: Combine Solutions

Now, combine the solutions of T(t) and X(x) to obtain the general solution for $\rho(x,t)$:

$$\rho(x,t) = T(t)X(x) = \left(C_1E_\alpha(\lambda t^\alpha) + C_2E_\alpha(-\lambda t^\alpha)\right)(C_3\cos kx + C_4\sin kx)$$

Step 5: Apply Initial or Boundary Conditions

$C_1, C_2, C_3, C_4, \lambda$ and k are constants determined by the initial or boundary conditions.

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Positive Solutions of Eigenvalue Problems for a Class of Fractional Differential Equations with Derivatives

in Abstract and Applied Analysis, 2012, Article ID 512127
https://doi.org/10.1155/2012/512127

By establishing a maximal principle and constructing upper and lower solutions, the existence of positive solutions for the eigenvalue problem of a class of fractional differential equations is discussed. Some sufficient conditions for the existence of positive solutions are established.

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The spectral analysis for a singular fractional differential equation with a signed measure

in Applied Mathematics and Computation 2015, 257: 252-263

In this paper, by using the spectral analysis of the relevant linear operator and Gelfand's formula, we obtain some properties of the first eigenvalue of a fractional differential equation. Based on these properties, the fixed point index of the nonlinear operator is calculated explicitly and some sufficient conditions for the existence of positive solutions are established.

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Nontrivial solutions for a fractional advection-dispersion equation in anomalous diffusion

in Applied Mathematics Letters 2017, 66:1-8

In this paper, we consider the existence of nontrivial solutions for a class of fractional advection–dispersion equations. A new existence result is established by introducing a suitable fractional derivative SOBOLEV space and using the critical point theorem.

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Iterative algorithm and estimation of solution for a fractional order differential equation

in Boundary Value Problem 2016, 1: 1-11.

In this paper, we establish an iterative algorithm and estimation of solutions for a fractional turbulent flow model in a porous medium under a suitable growth condition. Our main tool is the monotone iterative technique.

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The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model

in Nonlinear Analysis: Modelling and Control 2022, 27(3): 228-441.

In this paper, we consider the iterative properties of positive solutions for a general Hadamard-type singular fractional turbulent flow model involving a nonlinear operator. By developing a double monotone iterative technique we firstly establish the uniqueness of positive solutions for the corresponding model. Then we carry out the iterative analysis for the unique solution including the iterative schemes converging to the unique solution, error estimates, convergence rate and entire asymptotic behavior. In addition, we also give an example to illuminate our results.

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An upper-lower solution method for the eigenvalue problem of Hadamard-type singular fractional differential equation

in Nonlinear Analysis: Modelling and Control 2022, 27(4):789-802

In this paper, we are concerned with the eigenvalue problem of Hadamard-type singular fractional differential equations with multi-point boundary conditions. By constructing the upper and lower solutions of the eigenvalue problem and using the properties of the Green function, the eigenvalue interval of the problem is established via Schauder’s fixed point theorem. The main contribution of this work is on tackling the nonlinearity which possesses singularity on some space variables.

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Conclusion

- Existing Numerical approaches and limitations
- Recent Development methods
  - Variational iterative method: it is an influential method for solving linear and nonlinear problems.
  - Adomian decomposition method
  - Cubic B-spline functions for third-order time fractional differential equations.
  - Polynomial Algorithm: Legendre wavelet method, Legendre polynomials method, Bernstein polynomials method, Chebyshev wavelet method etc.

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Thank You...

https://ben-wiwat.github.io/

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